$$\mathrm{ch}\, x = \frac{e^x + e^{-x}}{2}, \quad \mathrm{sh}\, x = \frac{e^x - e^{-x}}{2}$$

$$\mathrm{sh}^2 x + 1 = \left(\frac{e^x - e^{-x}}{2}\right)^2 + 1 = \frac{e^{2x} - 2 + e^{-2x}}{4} + 1$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \mathrm{ch}^2 x$$

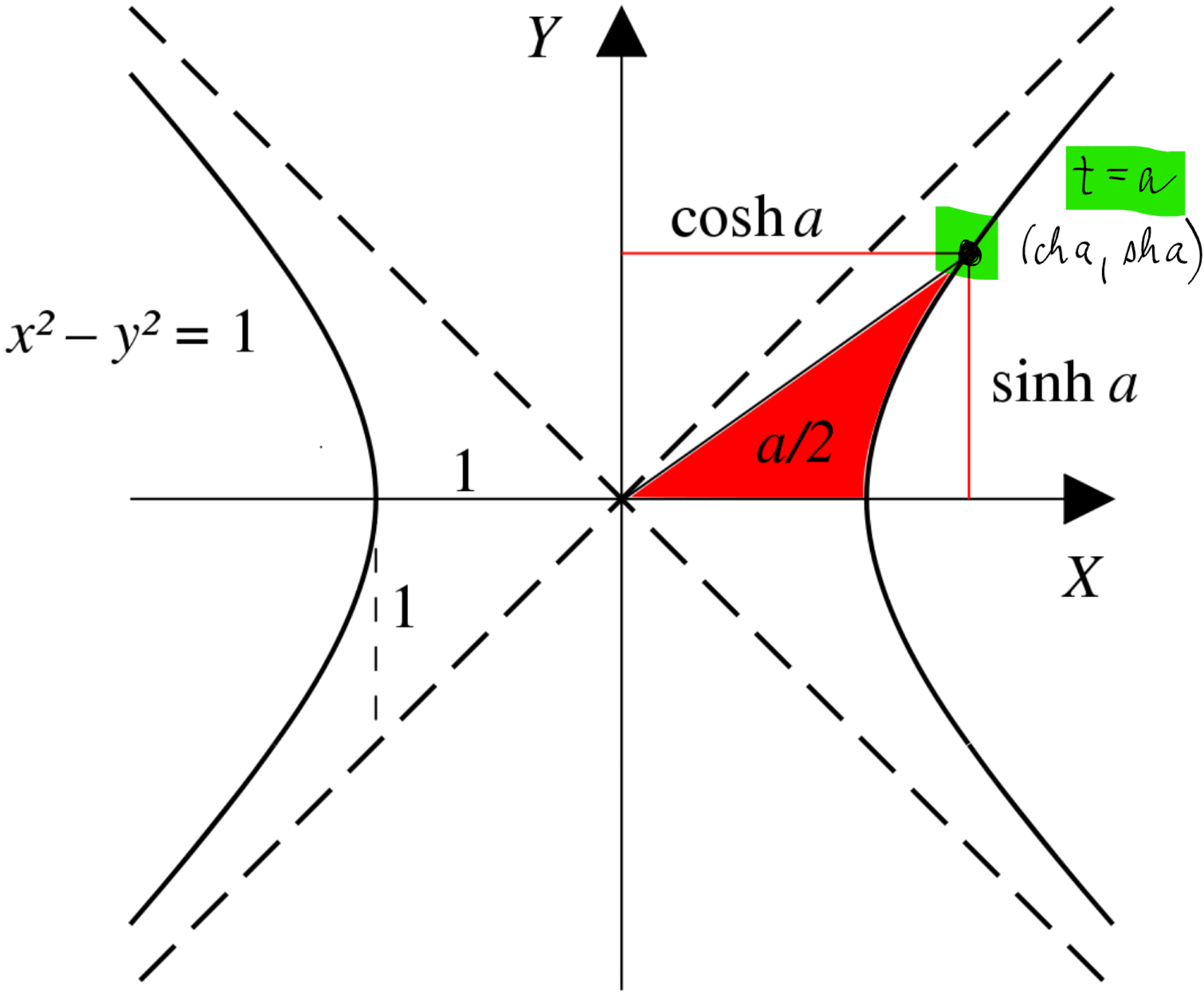
$\mathrm{sh}^2 x + 1 = \mathrm{ch}^2 x$ resp $\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$, $x \in \mathbb{R}$.

Křivka zadaná parametricky jako

$$\boxed{x = \mathrm{ch}\, t} \quad , t \in \mathbb{R}.$$
$$\boxed{y = \mathrm{sh}\, t}$$

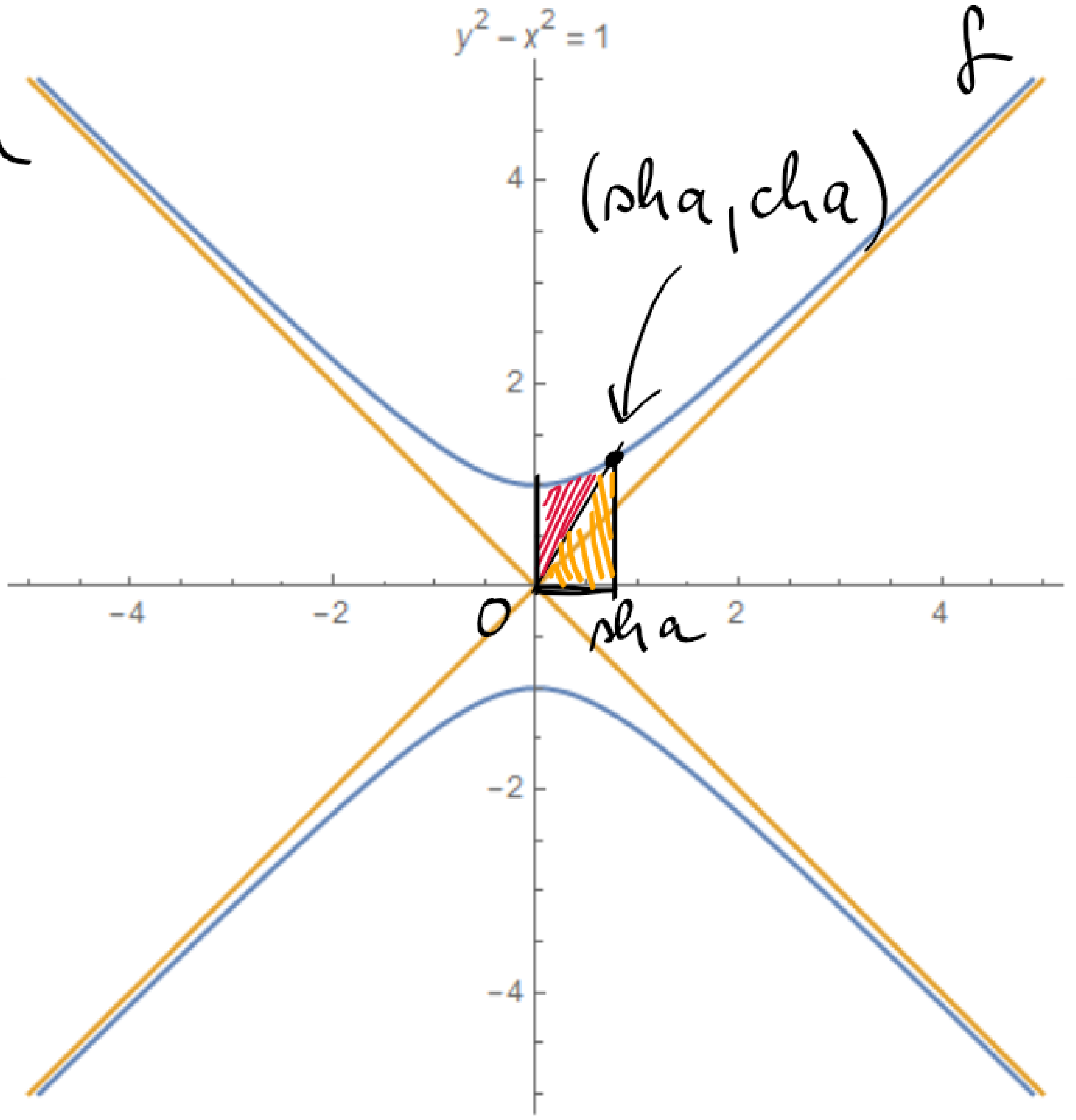
$$x^2 - y^2 \left(= \mathrm{ch}^2 t - \mathrm{sh}^2 t\right) = 1.$$

Vztah a (parametru) a červeně vybarvené plochy je patrný z obr.



Nastíjme nyní hyperbolu

$x = \operatorname{sh} t$
$y = \operatorname{ch} t \quad | \; t \in \mathbb{R}$



$y^2 - x^2 = 1$

$y^2 = 1 + x^2$

$y = \pm\sqrt{1+x^2}$ ... zajímá nás „+" (horní větev).

$f(x) = \sqrt{1+x^2}$. Platí: $S_{///} + S_{//} = \int_0^{\operatorname{sh} a} f(x)\,dx$.

ale $S_{//} = \frac{1}{2}\operatorname{sh} a \cdot \operatorname{ch} a$.

Tedy $S_{///} = \int_0^{\operatorname{sh} a} f(x)\,dx - \frac{1}{2}\operatorname{sh} a \cdot \operatorname{ch} a$.

Výpočet: $L = \int_0^{\operatorname{sh} a} \sqrt{1+x^2}\,dx$

N.-L.: $\int_0^{\operatorname{sh} a} f(x) = F(\operatorname{sh} a) - F(0)$, kde $F \in \int f$.

$\int \sqrt{1+x^2}\,dx$ $\qquad \operatorname{tg}: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{na} \mathbb{R}$, tj. $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\Rightarrow \cos y > 0$

a) SUBSTITUCE: $\left[\begin{array}{l} x = \operatorname{tg} y \quad (2.\text{VOS}) \\ dx = \frac{1}{\cos^2 y}\,dy \end{array}\right]$

$= \int \sqrt{1+\operatorname{tg}^2 y} \cdot \frac{1}{\cos^2 y}\,dy =$ $\qquad 1+\operatorname{tg}^2 y = 1 + \frac{\sin^2 y}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$

$= \int \sqrt{\frac{1}{\cos^2 y}} \cdot \frac{1}{\cos^2 y}\,dy =$

$= \int \frac{1}{|\cos y|} \cdot \frac{1}{\cos^2 y}\,dy = \int \frac{1}{\cos^3 y}\,dy =$

$= \int \frac{\cos y\,dy}{(1-\sin^2 y)^2} = \left| \begin{array}{l} \sin y = z \\ \cos y\,dy = dz \end{array} \right| = \int \frac{dz}{(1-z^2)^2} =$

$= \int \frac{dz}{(1-z)^2 \cdot (1+z)^2}$

$$\frac{1}{(1-z)^2(1+z)^2} = \frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{1+z} + \frac{D}{(1+z)^2} =$$

$$= \frac{1}{(1-z)^2(1+z)^2}\cdot\Big(A\cdot\underbrace{(1-z)(1+z)^2}_{(1-z)(1+2z+z^2)} + B\underbrace{(1+z)^2}_{(1+2z+z^2)} + C\underbrace{(1-z)^2(1+z)}_{(1-2z+z^2)(1+z)} + D(1-z)^2\Big)$$

$$1+2z+z^2-z-2z^2-z^3 \quad\Big|\quad z^2+2z+1 \quad\Big|\quad 1-2z+z^2+z-2z^2+z^3$$

$$1+z-z^2-z^3 \quad\Big|\quad \quad\Big|\quad 1-z-z^2+z^3$$

z^3: $-A+C=0$

z^2: $-A+B-C+D=0$

z: $A+2B-C-2D=0$

1: $A+B+C+D=1$

$$\left(\frac{1}{1-z}\right)' = \frac{-(-1)}{(1-z)^2} = \cancel{\ldots}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -3 & -1 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -3 & -1 \\ 0 & 0 & 4 & 8 & 3 \\ 0 & 0 & 4 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -3 & -1 \\ 0 & 0 & 4 & 8 & 3 \\ 0 & 0 & 0 & -4 & -1 \end{array}\right)$$

$D = \frac{1}{4}$ $\qquad 4C + 2 = 3$ $\qquad C = \frac{1}{4}$

$B - \frac{2}{4} - \frac{3}{4} = -1$ $\qquad B = \frac{1}{4}$

$A + 3/4 = 1$ $\qquad A = \frac{1}{4}$

$z = \sin y$

$x = \operatorname{tg} y$

$y = \operatorname{arctg} x$

$$\int \frac{1}{\ldots} = \frac{1}{4}\left(\int\frac{dz}{1-z} + \int\frac{dz}{(1-z)^2} + \int\frac{dz}{1+z} + \int\frac{dz}{(1+z)^2}\right)$$

$$= \frac{1}{4}\cdot\left(-\ln|1-z| + \frac{1}{1-z} + \ln|1+z| - \frac{1}{1+z}\right)$$

$$= \frac{1}{4}\left(\ln\left|\frac{1+z}{1-z}\right| + \frac{2z}{1-z^2}\right) =$$

$$= \frac{1}{4}\left(\ln\left|\frac{1+\sin y}{1-\sin y}\right| + \frac{2\sin y}{\cos^2 y}\right) = \frac{1}{4}\ln\left|\frac{1+\sin\operatorname{arctg}x}{1-\sin\operatorname{arctg}x}\right| + \ldots$$

$\bullet \sin(\operatorname{arctg} x)$ | $\sin y$, $\quad y = \operatorname{arctg} x$

$x = \operatorname{tg} y$

$\left(y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$$\operatorname{tg}^2 y = \frac{\sin^2 y}{\cos^2 y} = \frac{\sin^2 y}{1-\sin^2 y}$$

$$\Rightarrow \operatorname{tg}^2 y\,(1-\sin^2 y) = \sin^2 y$$

$$\operatorname{tg}^2 y - \operatorname{tg}^2 y \sin^2 y = \sin^2 y$$

$$\operatorname{tg}^2 y = \sin^2 y\,(1+\operatorname{tg}^2 y)$$

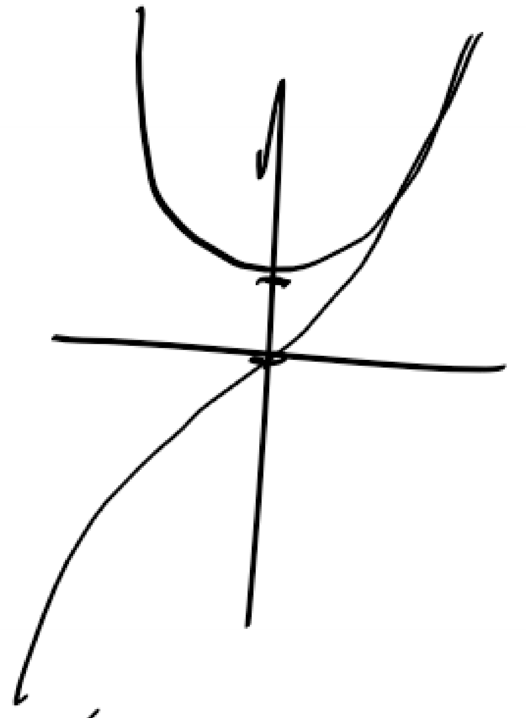
$$\frac{\operatorname{tg}^2 y}{1+\operatorname{tg}^2 y} = \sin^2 y \Rightarrow \cos^2 y = 1-\sin^2 y = 1-\frac{\operatorname{tg}^2 y}{1+\operatorname{tg}^2 y} = \frac{1}{1+\operatorname{tg}^2 y}$$

$$\Rightarrow \sin y = \frac{\operatorname{tg} y}{\sqrt{1+\operatorname{tg}^2 y}}, \quad y \in D_{\operatorname{tg}}$$

$$\sin(\operatorname{arctg} x) = \frac{\operatorname{tg} \operatorname{arctg} x}{\sqrt{1+\operatorname{tg}^2 \operatorname{arctg} x}} = \frac{x}{\sqrt{1+x^2}}$$

Analogicky $\cos(\operatorname{arctg} x) = \dfrac{1}{1+x^2}$.

b) $\displaystyle\int \sqrt{1+x^2}\,dx =$



Uvědomme si, že $\operatorname{ch}^2 y - \operatorname{sh}^2 y = 1$

Tedy $\operatorname{ch}^2 y = 1 + \operatorname{sh}^2 y \Rightarrow$ NABÍZÍ SE (2. VOS) SUBSTITUCE

$$= \left| \begin{array}{l} x = \operatorname{sh}^2 y \\ dx = 2 \operatorname{sh} y \operatorname{ch} y\, dy \end{array} \right| =$$

$$= \int \sqrt{1+\operatorname{sh}^2 y} \cdot 2 \operatorname{sh} y \operatorname{ch} y\, dy = \int \sqrt{\operatorname{ch}^2 y} \cdot 2 \operatorname{sh} y \operatorname{ch} y\, dy$$

$$= \int \operatorname{ch} y \cdot 2 \operatorname{sh} y \operatorname{ch} y\, dy = 2\int \operatorname{ch}^2 y \operatorname{sh} y\, dy =$$

$$\left| \begin{array}{l} z = \operatorname{ch} y \\ dz = \operatorname{sh} y\, dy \end{array} \right| = 2\int z^2\, dz = \frac{2}{3} z^3 =$$

$$= \frac{2}{3} \operatorname{ch}^3 y$$

$$\left| \begin{array}{l} x = \operatorname{sh} y \\ dx = \operatorname{ch} y \, dy \end{array} \right| = \int \sqrt{1+\operatorname{sh}^2 y}\; \operatorname{ch} y \, dy =$$

$$= \int \operatorname{ch}^2 y \, dy = \int \left(\frac{e^{y} + e^{-y}}{2} \right)^2 dy =$$

$$= \frac{1}{4} \int \left(e^{2y} + 2 + e^{-2y} \right) dy \overset{c}{=}$$

$$\overset{c}{=} \frac{1}{4} \cdot \left(\frac{1}{2} e^{2y} + 2y - \frac{1}{2} e^{-2y} \right) =$$

$$y = \operatorname{argsh} x = \ln\left(x + \sqrt{1+x^2}\right) \quad , \; x \in \mathbb{R}$$

$$= \frac{1}{4} \left(\frac{1}{2} e^{2 \ln\left(x+\sqrt{1+x^2}\right)} + 2 \ln\left(x+\sqrt{1+x^2}\right) - \frac{1}{2} e^{-2 \ln\left(x+\sqrt{1+x^2}\right)} \right) =$$

$$= \frac{1}{8} \left(x + \sqrt{1+x^2} \right)^2 + \frac{1}{2} \underbrace{\ln\left(x + \sqrt{1+x^2} \right)}_{\operatorname{argsh} x} - \frac{1}{8} \left(x + \sqrt{1+x^2} \right)^{-2}$$

$\sin 2\alpha = 2 \sin\alpha \cos\alpha$

$\operatorname{sh} 2\alpha = 2 \operatorname{sh}\alpha \operatorname{ch}\alpha$

$$\int_0^{\operatorname{sh} a} \sqrt{1+x^2}\, dx = \frac{1}{8} \left(\operatorname{sh} a + \sqrt{1+\operatorname{sh}^2 a} \right)^2 + \frac{1}{2} a - \frac{1}{8} \left(\operatorname{sh} a + \sqrt{1+\operatorname{sh}^2 a} \right)^{2}$$

Chceme: $\frac{1}{2} \operatorname{sh} a \cdot \operatorname{ch} a$

$$\left(\operatorname{sh} a + \sqrt{1+\operatorname{sh}^2 a} \right)^2 = \left(\operatorname{sh} a + \operatorname{ch} a \right)^2 = \operatorname{sh}^2 a + 2 \operatorname{sh} a \operatorname{ch} a + \operatorname{ch}^2 a$$

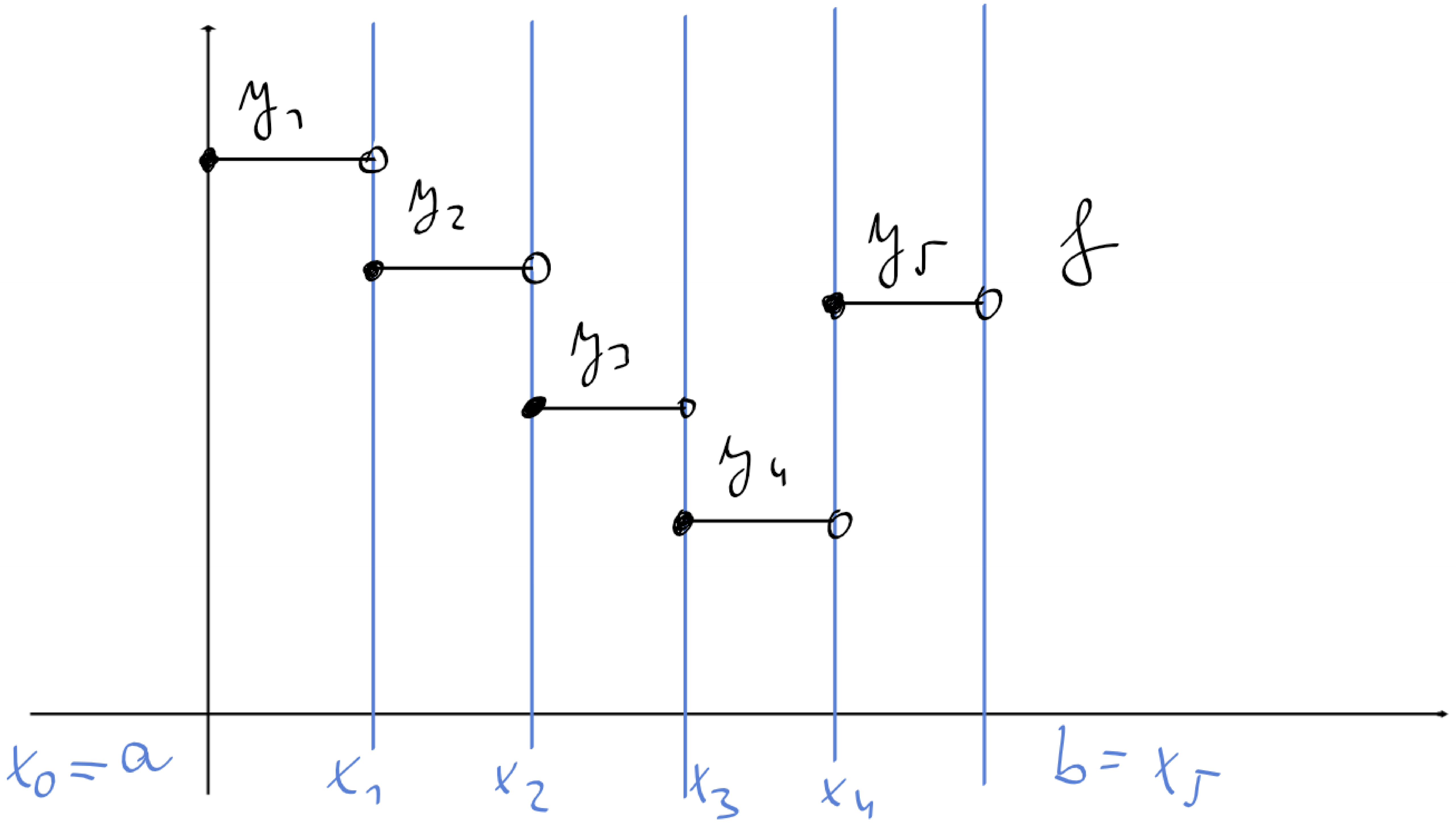
$$\left(\text{— ″ —} \right)^{-2} = \frac{1}{\left(\operatorname{sh} a + \operatorname{ch} a \right)^2}$$

$\operatorname{ch}^2 a - \operatorname{sh}^2 a$ ↓

$$\left(\operatorname{sh} a + \operatorname{ch} a \right)^2 - \frac{1}{\left(\operatorname{sh} a + \operatorname{ch} a \right)^2} = \frac{\overbrace{\left(\operatorname{sh} a + \operatorname{ch} a \right)}^{e^a}{}^4 - 1}{\left(\operatorname{sh} a + \operatorname{ch} a \right)^2} =$$

$$z^2 - \frac{1}{z^2} = \frac{z^4 - 1}{z^2} = \frac{(z^2-1)(z^2+1)}{z^2} = \dots$$

Průměrná hodnota funkce:

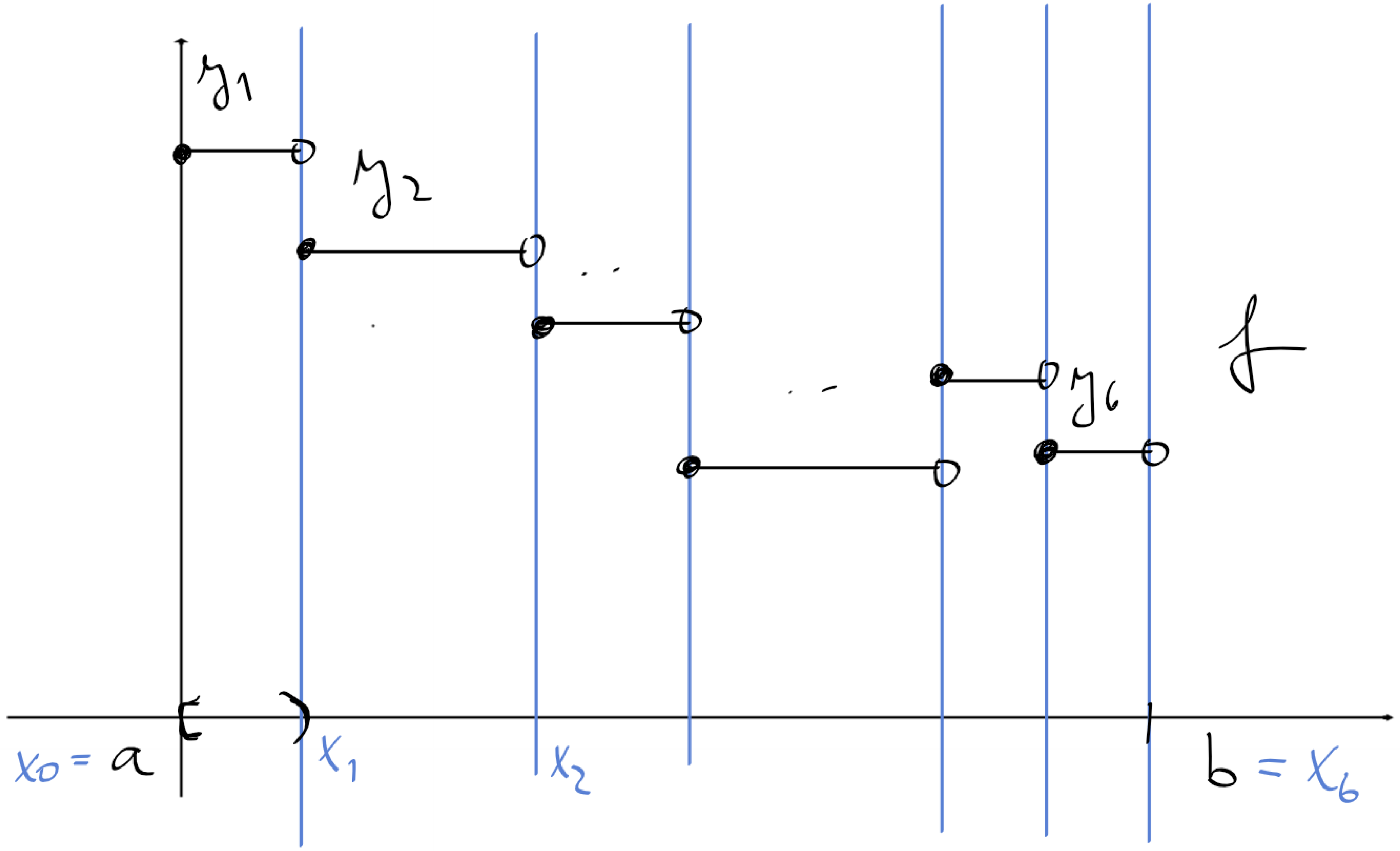


$|x_i - x_{i-1}| = c$, $i = 1, \dots, 5$ (int. mají délku c)

$$P(f, [a,b]) = \frac{1}{5}(y_1 + y_2 + y_3 + y_4 + y_5)$$

Pokud máme nestejně dlouhé dělící int.:

Vážený průměr:



$$P(f, [a,b]) = \sum_{i=1}^{6} \frac{x_i - x_{i-1}}{b-a} \cdot y_i =$$

$$= \frac{1}{b-a} \sum_{i=1}^{6} (x_i - x_{i-1}) \cdot y_i = \frac{1}{b-a} \underbrace{\sum_{i=1}^{6} (x_i - x_{i-1}) f(x_{i-1})}_{\text{integrální součet}}$$